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UNSTEADY MOTION OF A THIN WING IN A STRATIFIED FLUID

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Although there have been many studies of the motion of a rigid body in a stratified fluid [1-3], they have dealt with irrotational flow about symmetrical bodies: cylinders, spheres, etc.

In this investigation, we examine the two-dimensional problem of flow about a wing with a sharp edge with allowance for the trailing vortex. As in a uniform fluid, the vortex behind the wing is replaced by a line of tangential velocity discontinuity.

We formulate an initial-boundary-value problem for the stream function, and we seek the solution in the form of a logarithmic dynamic potential [4] with an unknown density. To determine the density of the potential, we obtain a singular integral equation for the plate and relations for the vortex. It is shown that these equations have solutions at high Froude numbers. Here, the classical Zhukov formula for buoyancy remains valid for a thin wing in a stratified fluid. We solve the problem numerically and study the effect of the initial parameters on the steady-state buoyancy and moment and nonsteady hydrodynamic forces acting on the wing.

The numerical solution and the asymptote obtained at high Froude numbers show that the buoyancy of the wing in a stratified fluid depends nonmonotonically on the Froude number. Buoyancy in the stratified fluid is lower than in a uniform fluid at high Froude numbers, while the opposite is true at low Froude numbers. The moment approaches zero with a decrease in the Froude number, i.e., the center of pressure shifts toward the middle of the wing. These results show that the presence of stratification can have a significant effect on the hydrodynamic characteristics of a wing.

1. Small two-dimensional perturbations of an incompressible inviscid fluid in a cartesian coordinate system (x_1, x_2) are described by the system of equations

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} + \operatorname{grad} p + \mathbf{e}_2\, g\, \rho_1 = 0, \quad \frac{\partial \rho_1}{\partial t} + \rho_0'(x_2) v_2 = 0, \quad \operatorname{div} \mathbf{v} = 0, \tag{1.1}$$

where $\mathbf{v} = (v_1, v_2)$ is the velocity vector; $\rho_0(x_2)$ is density in the undisturbed state; ρ_1 is the density perturbation; p is perturbed pressure; e_2 is a unit vector on the Ox_2 axis; g is acceleration due to gravity. It is assumed that the stratification is slight and exponential in nature:

$$\rho_0(x_2) = \rho_* \exp(-\beta x_2), \quad \beta = \text{const} \ll 1.$$

Introducing the stream function $\psi(x)$, we use system (1.1) to obtain the equation

$$\frac{\partial^2}{\partial t^2}\left(\Delta \psi - \beta \frac{\partial \psi}{\partial x_2}\right) + g\, \beta \frac{\partial^2 \psi}{\partial x_1^2} = 0.$$

We will assume that the dimensions of the wing are small compared to the characteristic dimension of the stratification, i.e., $b \ll 1/\beta$ (b is the chord of the wing). We can then use the Boussinesq approximation and ignore the term $\beta \psi'_{x_2}$. As a result, we obtain the Sobolev equation:

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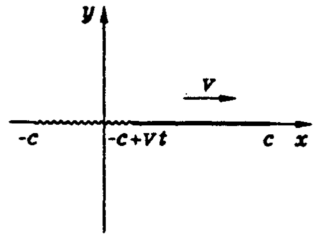

Fig. 1

$$\frac{\partial^2}{\partial t^2}\Delta\psi + \omega_0^2 \frac{\partial^2 \psi}{\partial x_1^2} = 0 \tag{1.2}$$

($\omega_0^2 = -g\rho'_0/\rho_0$ is the Brent–Weizall frequency).

The wing will be modeled as an infinitely thin plate of length 2c (c is the half-chord of the wing, which is positioned horizontally and moves at a constant velocity V). Here, the liquid remains undisturbed. At the moment of time t = 0 the wing occupies the segment $l_0 = (-c, c)$ of the Ox_1 axis and begins to undergo small vibrations in accordance with an assigned law

$$v_2(\sigma, t) = f(\sigma, t) \tag{1.3}$$

(σ is the arc coordinate on the wing).

At the moment of time t = 0, a vortex sheet also begins to be shed from the wing. As in the case of a uniform fluid, the vortex sheet will be modeled by a line of tangential velocity discontinuity. At a certain moment of time t, the wing will occupy the position $l_{0t} = (-c + Vt, c + Vt)$, while the position of the vortex sheet will be $l_{1t} = (-c, -c + Vt)$ (Fig. 1).

Thus, the motion of the fluid satisfies Eq. (1.2) outside the contour l_t at t > 0 and the initial conditions

$$\psi(\mathbf{x}, 0) = \psi_t(\mathbf{x}, 0) = 0, \tag{1.4}$$

as well as the following boundary conditions: the condition of nonflow on the wing (1.3), conditions on the line of tangential discontinuity l_{1t}

$$[v_2] = 0, \qquad [p] = 0, \tag{1.5}$$

and the Kutta–Zhukov conditions at the trailing edge. The brackets denote the difference between the limiting values above and below the contour.

In addition, the function $\psi(\mathbf{x}, t)$ must satisfy the conditions of regularity at infinity:

$$|\nabla\psi| = O(r^{-2}), \quad r = \sqrt{x_1^2 + x_2^2}, \quad r \to \infty, \tag{1.6}$$

as well as the condition on the leading edge

$$|\nabla\psi| = O(r_1^{-\delta}), \quad 0 < \delta < 1 \tag{1.7}$$

(r_1 is the distance to the leading edge). In physical terms, regularity conditions (1.6) and (1.7) mean that the energy is finite.

2. A logarithmic dynamic potential was introduced in [4] for Eq. (1.2). In the general case, this potential is valid only for a stationary boundary. However, it is valid for the case we are considering – a wing with a wake. We will seek the solution of problem (1.2)-(1.7) in the form of a logarithmic dynamic potential

$$\psi(\mathbf{x},t) = \int\limits_{l_t} \mu(\xi,t) \ln|\mathbf{x}-\mathbf{y}(\xi)|\, d\xi + \int\limits_0^t d\sigma \int\limits_{l_\sigma} \mu(\xi,\sigma) \frac{1}{t-\sigma} \left[1 - \cos\left(\omega_0(t-\sigma)\frac{x_2 - y_2(\xi)}{|\mathbf{x}-\mathbf{y}(\xi)|}\right)\right] d\xi, \quad (2.1)$$

where $\mu(\xi, t)$ is the unknown density; $l_t = l_{0t} \cup l_{1t}$.

The function $\psi(\mathbf{x}, t)$, determined by Eq. (2.1), satisfies condition (1.2) outside the contour l_t. This can be verified directly. The rear point of the contour l_t is stationary and free of singularities, while infinity of velocity in the form (1.7) is allowed at the forward point.

It follows from (1.4) that

$$\mu(\xi,0) = \mu_t(\xi,0) = 0. \quad (2.2)$$

The first condition of (1.5) is satisfied by virtue of representation (2.1). Let us consider the second condition. We calculate the discontinuity of tangential velocity on the contour l_t. We find from (2.1) that

$$v_1(\mathbf{x},t) = \int\limits_{l_t} \mu(\xi,t) \frac{x_2}{(x_1-\xi)^2 + x_2^2}\, d\xi +$$
$$+\omega_0 \int\limits_0^t d\sigma \int\limits_{-c}^{c+V\sigma} \mu(\xi,\sigma) \frac{(x_1-\xi)^2}{[(x_1-\xi)^2 + x_2^2]^{3/2}} \sin\left(\omega_0 \frac{(t-\sigma)x_2}{\sqrt{(x_1-\xi)^2 + x_2^2}}\right) d\xi.$$

The first integral is the normal derivative of the potential of a simple layer [5], while in the second integral we make the substitution of variables $\xi - x_1 = \eta\,|x_2|$. At $|x_2| \to 0$, for the discontinuity of tangential velocity $[v_1(\xi, t)] = v_1(\xi_1 + 0, t) - v_1(\xi_1 - 0, t)$ we obtain

$$[v_1(\xi,t)] = \begin{cases} 2\pi[\mu(\xi,t) + \omega_0 \int\limits_{a(\xi)}^t \mu(\xi,\sigma) G(\omega_0(t-\sigma))\, d\sigma], & -c < \xi < c + Vt, \\ 0, \quad \xi < -c, \quad \xi > c + Vt, \end{cases} \quad (2.3)$$

where $a(\xi) = \max\{0, (\xi - c)/V\}$ is the moment of time at which a velocity discontinuity develops at point ξ;

$$G(\omega_0(t)) = \int\limits_{-\infty}^{\infty} \frac{\eta^2}{(1+\eta^2)^{3/2}} \sin\left(\frac{\omega_0 t}{\sqrt{1+\eta^2}}\right) d\eta = \int\limits_0^{\omega_0 t} \frac{J_1(x)}{x}\, dx.$$

It follows from the first equation of system (1.1) that the following relation is satisfied on the contour l_t

$$\frac{\partial [v_1]}{\partial t} = -\frac{1}{\rho_0} \frac{\partial}{\partial x_1} [p]. \quad (2.4)$$

Since $[p] = 0$ in the wake, the discontinuity of tangential velocity in a fluid particle in the wake does not change — as in a uniform fluid:

$$[v_1(\xi,t)] = \left[v_1\left(\xi, \frac{\xi + c}{V}\right)\right] \quad \text{at} \quad t > \frac{\xi + c}{V}. \quad (2.5)$$

Here, $(\xi + c)/V$ is the moment of time at which the trailing edge of the wing passes the point ξ. Inserting Eq. (2.3) into (2.5), we obtain

$$\mu(\xi,t) + \omega_0 \int\limits_{a(\xi)}^t \mu(\xi,\sigma) G(\omega_0(t-\sigma))\, d\sigma =$$
$$= \mu\left(\xi, \frac{\xi+c}{V}\right) + \omega_0 \int\limits_{a(\xi)}^{\frac{\xi+c}{V}} \mu(\xi,\sigma) G\left(\omega_0\left(\frac{\xi+c}{V} - \sigma\right)\right) d\sigma. \quad (2.6)$$

Fig. 2

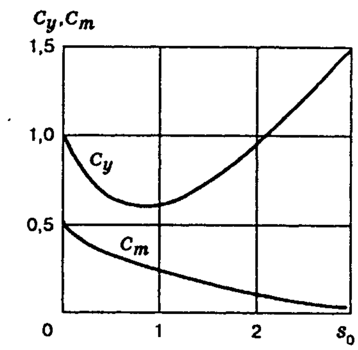

Fig. 3

We designate $\tau = t - (\xi + c)/V$. We then represent (2.6) in the form

$$\mu\left(\xi,\frac{\xi+c}{V}+\tau\right)+\omega_0\int\limits_0^{\tau}\mu\left(\xi,\sigma+\frac{\xi+c}{V}\right)G\left(\omega_0(\tau-\sigma)\right)d\sigma=\mu\left(\xi,\frac{\xi+c}{V}\right)+$$
$$+\omega_0\int\limits_{a(\xi)}^{\frac{\xi+c}{V}}\mu(\xi,\sigma)G\left(\omega_0\left(\frac{\xi+c}{V}-\sigma\right)\right)d\sigma-\omega_0\int\limits_{a(\xi)}^{\frac{\xi+c}{V}}\mu(\xi,\sigma)G\left(\omega_0(t-\sigma)\right)d\sigma. \quad (2.7)$$

We write the operator on the left side of integral equation (2.7) in the form $I + \omega_0 G^*$ (G^* is the operator for convolution with the function G and I is the unit operator). The inverse operator was found in [4]:

$$(I+\omega_0 G*)^{-1}=I+\omega_0 J_1* \quad (2.8)$$

(J_{1*} is the operator for convolution with the Bessel function $J_1(\omega_0\tau)$). We use (2.7) to obtain a representation for the density of potential in terms of density on the wing:

$$\mu(\xi,t)=\left[\mu\left(\xi,\frac{\xi+c}{V}\right)+\omega_0\int\limits_{a(\xi)}^{\frac{\xi+c}{V}}\mu(\xi,\sigma)G\left(\omega_0\left(\frac{\xi+c}{V}-\sigma\right)\right)d\sigma\right]J_0\left(\omega_0\left(t-\frac{\xi+c}{V}\right)\right)-$$
$$-\omega_0\int\limits_{a(\xi)}^{\frac{\xi+c}{V}}\mu(\xi,\sigma)\left[J_1(\omega_0(t-\sigma))+\omega_0\int\limits_0^{\frac{\xi+c}{V}-\sigma}J_1(\omega_0(t-\tau-\sigma))G(\omega_0\tau)\,d\tau\right]d\sigma. \quad (2.9)$$

The Berkness theorem is valid for a stratified fluid. In a linear approximation, this theorem has the form

$$\frac{d\Gamma}{dt}=-\oint\limits_l\frac{dp}{\rho_0}.$$

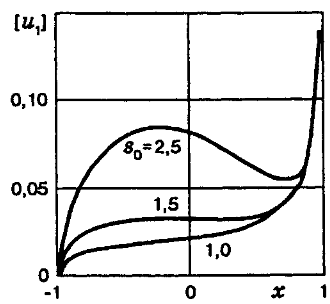

Fig. 4

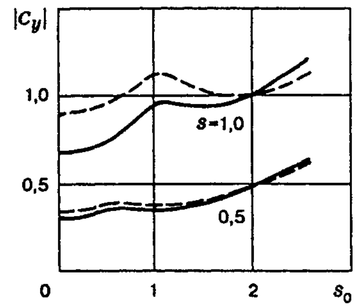

Fig. 5

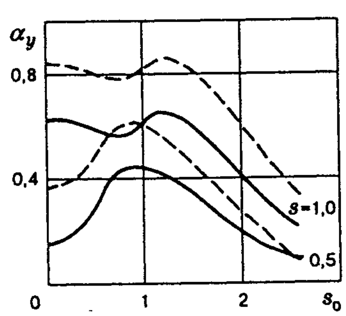

Fig. 6

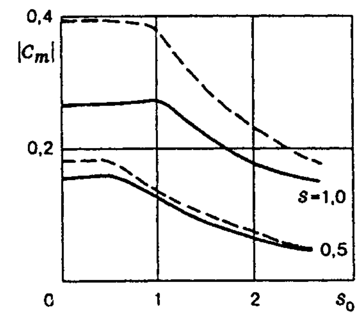

Fig. 7

Here, Γ(t) is the circulation of velocity over the fluid contour l. Applying it to the contour $l_t = l_{0t} + l_{1t}$ and considering that ρ_0 = const along l_t, we find that $d\Gamma/dt = 0$. We represent Γ(t) in the form $\Gamma(t) = \Gamma_0(t) + \Gamma_1(t)$, where $\Gamma_0(t)$ and $\Gamma_1(t)$ are the circulation of velocity about the contours l_{0t} and l_{1t}, respectively:

$$\Gamma_0 = \int\limits_{l_{0t}} [v_1(\xi, t)]\, d\xi, \qquad \Gamma_1(t) = \int\limits_{-c}^{-c+Vt} [v_1(\xi, t)]\, d\xi.$$

Now differentiating with respect to t and considering that $[v_{1t}] = 0$ in the wake, we find the relation

$$[v_1(-c + Vt, t)] = -\frac{1}{V}\frac{d\Gamma_0}{dt}(t), \tag{2.10}$$

which for the velocity discontinuity at the trailing edge has exactly the same form as in a uniform liquid.

Inserting (2.3) into (2.10), we obtain a representation for the density of potential on the trailing edge in terms of density on the wing:

$$\mu(-c+Vt, t) = -\frac{1}{V}\frac{d}{dt}\int\limits_{-c+Vt}^{c+Vt} \mu(\xi, t)\, d\xi - \omega_0 \int\limits_{a(t-2c/V)}^{t} \mu(-c+Vt, \sigma)\, G(\omega_0(t-\sigma))\, d\sigma -$$
$$-\frac{\omega_0}{V}\frac{\partial}{\partial t}\int\limits_{-c+Vt}^{c+Vt} d\xi \int\limits_{a(\xi)}^{t} \mu(\xi, \sigma)\, G(\omega_0(t-\sigma))\, d\sigma.$$

To satisfy the condition of regularity at infinity (1.6), it is necessary to also satisfy the condition

$$C(t) \equiv \int\limits_{l_t} \mu(\xi, t)\, d\xi = 0.$$

Changing the order of integration, we find from (2.3) that

$$\Gamma(t) = C(t) + \omega_0 \int_0^t C(t)\, G(\omega_0 (t-\sigma))\, d\sigma.$$

Since $\Gamma(t) \equiv 0$, it follows from (2.8) that $C(t) \equiv 0$.

Now let us change over to the coordinate system (X_1, X_2), which is connected with the wing by the formulas

$$x_1 = X_1 + Vt, \qquad x_2 = X_2.$$

Let us determine the stream function of the perturbed flow $\Psi(X, t) = \psi(x_1 + Vt, x_2, t)$. The wing is stationary in this coordinate system, $L_0 = (-c, c)$, while at the moment of time t the wake occupies the position $L_{1t} = (-c - Vt, -c)$, $L_t = L_0 \cup L_{1t}$. We put $\nu(\xi, t) = \mu(\xi + Vt, t)$. Then the following representation is valid for the stream function $\Psi(X, t)$

$$\Psi(X,t) = \int_{L_t} \nu(\xi,t) \ln |\mathbf{X} - \mathbf{Y}(\xi)|\, d\xi + \\ + \int_0^t d\sigma \int_{L_\sigma} \nu(s,\sigma) \frac{1}{t-\sigma} \left[1 - \cos\left(\omega_0 (t-\sigma) \frac{X_2}{|\mathbf{X} + V\mathbf{e}_1(t-\sigma) - \mathbf{Y}(\xi)|}\right)\right] d\xi. \tag{2.11}$$

It follows from (2.2) that

$$\nu(\xi,t) = \nu_t(\xi,t) = 0, \quad \xi \in L_t.$$

We use (2.3) to obtain the following for the discontinuity of tangential velocity on the contour L_t

$$[u_1(\xi,t)] = 2\pi \left[\nu(\xi,t) + \frac{\omega_0}{V} \int_\xi^c \nu\left(\eta, t + \frac{\xi-\eta}{V}\right) \theta\left(t + \frac{\xi-\eta}{V}\right) G\left(\omega_0 \frac{\eta-\xi}{V}\right) d\eta\right], \tag{2.12}$$

where $\theta(t)$ is the Heaviside function.

Equation (2.9) takes the following form for the density of potential in the wake

$$\nu(\xi,t) = \gamma\left(t + \frac{\xi+c}{V}\right) J_0\left(\omega_0 \frac{\xi+c}{V}\right) + h(\xi,t), \quad \xi \in L_{1t}. \tag{2.13}$$

As in a uniform fluid, the function $\gamma(t)$ corresponds physically to the discontinuity of velocity at the trailing edge

$$\gamma(t) = [u_1(-c,t)] = -\frac{1}{V} \frac{d\Gamma}{dt}$$

and is calculated through the density on the wing by means of the formula

$$\gamma(t) = -\frac{1}{V} \frac{d}{dt} \int_{-c}^{c} \nu(\eta,t)\, d\eta - \\ - \frac{\omega_0}{V^2} \frac{d}{dt} \int_{-c}^{c} d\eta \int_{-c}^{\eta} \nu\left(\eta, t + \frac{\xi-\eta}{V}\right) \theta\left(t + \frac{\xi-\eta}{V}\right) G\left(\omega_0 \frac{\xi-\eta}{V}\right) d\xi. \tag{2.14}$$

The function $h(\xi, t)$ is the complement due to stratification:

$$h(\xi,t) = -\frac{\omega_0}{V} \int_{-c}^{c} \nu\left(\eta, t + \frac{\xi-\eta}{V}\right) \left[J_1\left(\omega_0 \frac{\eta-\xi}{V}\right) + \omega_0 \int_0^{\frac{\eta+c}{V}} J_1\left(\omega_0 \left(\frac{\eta-\xi}{V} - \tau\right)\right) G(\omega_0 \tau)\, d\tau\right] d\eta. \tag{2.15}$$

It follows from the properties of the logarithmic dynamic potential [4] that the condition of nonflow on the wing assumes a very simple form:

$$\int_{L_t} \frac{\nu(\xi,t)}{\xi - x}\, d\xi = f(x,t), \quad x \in L_0. \tag{2.16}$$

Thus, for the unknown density of potential $\nu(\eta, t)$ on the wing, we have singular integral equation (2.16). We have Eqs. (2.13-2.15) for the density of potential in the wake. It should be noted that all of the formulas obtained above become the corresponding relations for a uniform fluid when $\omega_0 = 0$.

3. It is more convenient for the purposes of our investigation if we reduce system (2.13)-(2.16) to one of Fredholm's integral equations. We introduce the auxiliary analytic function

$$\Phi(z,t) = \frac{1}{2\pi i} \int_{L_t} \frac{\nu(\xi,t)}{\xi - z}\, d\xi.$$

It follows from (2.16) that it satisfies the conditions of a Riemannian problem:

$$\Phi^+(\xi,t) = \Phi^-(\xi,t) + \nu(\xi,t), \qquad \xi \in L_{1t},$$

$$\Phi^+(\xi,t) = -\Phi^-(\xi,t) - i\pi f(\xi,t), \quad \xi \in L_0.$$

Considering conditions (1.7) and the Zhukov condition, we seek the solution in the class of functions that are unbounded on the leading edge but bounded on the trailing edge. Solving this problem [6], we obtain an expression for the density of potential on the wing in terms of the density of potential in the wake:

$$\nu(s,t) = -\frac{1}{\pi}\sqrt{\frac{c+s}{c-s}}\left(\int_{L_{1t}} \sqrt{\frac{\xi - c}{\xi + c}}\, \frac{\nu(\xi,t)}{\xi - s}\, d\xi + \frac{1}{\pi}\int_{L_0} \sqrt{\frac{c-\xi}{c+\xi}}\, \frac{f(\xi,t)}{\xi - s}\, d\xi \right).$$

Inserting Eqs. (2.13)-(2.15) for density in the wake, we obtain an integral equation for density on the wing that includes time as a parameter. We use the Laplace transform on functions $\nu(\xi, t)$ and $f(\xi, t)$:

$$\nu^L(\xi,p) = \int_0^\infty e^{-pt}\nu(\xi,t)\, dt.$$

We then find a Fredholm integral equation of the second kind for the function $æ(\xi, p) = \nu^L(c\xi, p)$:

$$æ(\xi,p) + \frac{1}{\pi}\sqrt{\frac{1+\xi}{1-\xi}} \int_{-1}^{1} æ(\eta,p)\, T(\xi,\eta;z,s_0)\, d\eta = -\frac{1}{\pi^2}\sqrt{\frac{1+\xi}{1-\xi}} \int_{-1}^{1} \sqrt{\frac{1-\eta}{1+\eta}}\, \frac{f^L(c\eta,p)}{\eta - \xi}\, d\eta. \tag{3.1}$$

Here, the kernel $T(\xi, \eta; z, s_0)$ depends on two dimensionless parameters: on $z = pc/V$, which is the spectral parameter in the given problem; on the real parameter $s_0 = \omega_0 c/V$, which corresponds to the inverse of the Froude number. After the substitution $\xi = -\mathrm{ch}\,\lambda$, we can write the kernel T in the form

$$T(\xi,\eta;z,s_0) = ze^z\left[1 + s_0\int_0^{1+\eta} e^{-z\zeta} G(s_0\zeta)\, d\zeta\right] \int_0^\infty \frac{(1+\mathrm{ch}\lambda)e^{-z\,\mathrm{ch}\lambda} J_0(s_0(\mathrm{ch}\lambda - 1))}{\mathrm{ch}\lambda + \xi}\, d\lambda +$$

$$+ s_0 \int_0^\infty \frac{(1+\mathrm{ch}\lambda)e^{-z\,\mathrm{ch}\lambda}\left(J_1(s_0(\eta + \mathrm{ch}\lambda)) + s_0\int_0^{1+\eta} J_1(s_0(\eta + \mathrm{ch}\lambda - x))\, G(s_0 x)\, dx\right)}{\mathrm{ch}\lambda + \xi}\, d\lambda.$$

Inserting integral representations for the functions J_1 and J_0 into the above equation and considering that

$$\left(\frac{d}{dz}-\xi\right)\int\limits_0^\infty \frac{e^{-z\,\mathrm{ch}\lambda}}{\mathrm{ch}\lambda+\xi}\,d\lambda = -K_0(z),$$

where K_0 is a MacDonald function, we find the analytic continuation of T over the entire complex plane:

$$T(\xi,\eta;z,s_0)=\frac{ze^z}{\pi}\left[1+s_0\int\limits_0^{1+\eta} e^{-z\zeta}G(s_0\zeta)\,d\zeta\right]\times$$
$$\times\int\limits_0^\pi e^{-is_0\cos\theta}\left\{K_0(z-is_0\cos\theta)+(1-\xi)e^{\xi(z-is_0\cos\theta)}\left[\frac{2}{\sqrt{1-\xi^2}}\,\mathrm{arctg}\sqrt{\frac{1-\xi}{1+\xi}}-\right.\right.$$
$$\left.\left.\int\limits_0^{z-is_0\cos\theta} e^{-\xi y}K_0(y)\,dy\right]\right\}d\theta+\frac{s_0}{i\pi}\int\limits_0^\pi \cos\theta\, e^{-(z-is_0\cos\theta)\eta}\times \tag{3.2}$$
$$\times\left[1+s_0\int\limits_0^{1+\eta} e^{-is_0x\cos\theta}G(s_0x)\,dx\right]\left\{K_0(z-is_0\cos\theta)+(1-\xi)e^{\xi(z-is_0\cos\theta)}\times\right.$$
$$\left.\times\left[\frac{2}{\sqrt{1-\xi^2}}\,\mathrm{arctg}\sqrt{\frac{1-\xi}{1+\xi}}-\int\limits_0^{z-is_0\cos\theta} e^{-\xi y}K_0(y)\,dy\right]\right\}d\theta.$$

Since the function K_0 has a logarithmic singularity at zero, then T is analytic with respect to z throughout the complex plane except for a segment on the imaginary axis $[-is_0, is_0]$. It is also continuous with respect to z and s_0 throughout the complex plane.

To remove the singularity at the point $\xi = c$, we make the substitution of variables $\xi = \cos\alpha$ and we introduce the new unknown function $æ_1(\alpha, p) = \sqrt{1-\cos\alpha}\, æ(\cos\alpha, p)$. We then obtain a Fredholm equation with a continuous kernel on the interval of integration $\alpha \in [0, \pi]$:

$$(I+T_1)æ_1 = f_1; \tag{3.3}$$
$$T_1(\alpha,\beta;z,s_0)=2T(\cos\alpha,\cos\beta;z,s_0)\cos(\alpha/2)\cos(\beta/2)/\pi,$$
$$f_1(\alpha,p)=\sqrt{1-\cos\alpha}\,f^L(c\cos\alpha,p). \tag{3.4}$$

Let us examine Eq. (3.3) in the class of functions $L_2[0, \pi]$. We will show that Eq. (3.3) does not have discrete spectrum points for a weakly stratified fluid ($s_0 << 1$). At $z = O(s_0)$, the operator T_1 from (3.2), (3.4) is compressible, $\|T_1\| = O(s_0\,|\ln(s_0)|)$.

In the case $|z| >> s_0$, the principal part of the operator T at $\mathrm{Re}\, z \geq 0$ is independent of the variable of integration η and is thus easily inverted. As a result, we obtain the equation

$$(I+s_0^2T_2)æ_1 = f_2,$$

where the norm of the operator T_2 is finite. It follows from this that with small s_0, Eq. (3.1) does not have discrete spectrum points at $\mathrm{Re}\, z \geq 0$, $|z| >> s_0$. It follows from theorem 7.2 [7, p. 382] that with an increase in s_0, discrete spectrum points can appear only on the boundary of the region in which the kernel is analytic, i.e., on the segment $[-is_0, is_0]$.

4. Let us examine the question of the asymptote of the solution at long times ($t \to \infty$) for three cases: 1) the function $f(\sigma, t)$ is finite over time; 2) $f(\sigma, t) = f_0(\sigma)e^{i\omega t}$ are harmonic vibrations of the wing with the frequency ω; 3) $f(\sigma, t) = f_0(\sigma)$, where $f_0(\sigma)$ is an assigned function corresponding (for example) to the motion of the wing at a small angle of attack.

In these cases, the right side of Eq. (3.3) will behave as follows. In the first case, the function $f_1^L(\alpha, p)$ is analytic with respect to p throughout the complex plane. In the second case, $f_1^L(\alpha, p)$ has a pole at $p = i\omega$. In the third case, $f_1^L(\alpha, p)$ has a pole at $p = 0$. Since Eq. (3.3) is unambiguously solvable at small s_0, then the solution $æ_1(\alpha, p)$ will behave with respect to p in the same manner as f_1^L, i.e. it will have one pole in the second and third cases and be analytic with respect to p in the first case.

Now we use the residue theorem to find the asymptote of the solution $æ_1(\alpha, t)$ at $t \to \infty$. We will show that on circle arcs $C_R(|z| = R, \mathrm{Re}\, z < a)$, $æ_1(\alpha, p) \to 0$ at $R \to \infty$ uniformly relative to the argument z. It follows from (3.2) that there exists $A > 0$ such that $\|T_1\| > A\sqrt{|z|}$ for $z \in C_R$. Then $\|(I + T_1)^{-1}\| < 1/(A\sqrt{|z|} - 1)$. Using the Jordan lemma [8], we obtain

$$\lim_{R\to\infty} \int_{C_R} æ_1(\alpha, p) e^{pt}\, dt = 0 \qquad \text{at} \qquad t > 0.$$

Since the kernel T_1 is continuous with respect to z on the segment $[-is_0, is_0]$, then at small s_0 the inverse operator $(I + T_1)^{-1}$ will also be continuous. It follows from this that in the first case $æ(\alpha, t) \to 0$ at $t \to \infty$, so that $\Psi(X, t) \to 0$, i.e. the perturbations decay over time. In the second case, $\Psi(X, t) \underset{t\to\infty}{\to} 0$, i.e. a steady vibrational regime $\nu(x, t) \underset{t\to\infty}{\to} \nu_*(x)e^{i\omega t}$, is seen in the coordinate system connected with the wing. In the third case, $\Psi(X, t) \underset{t\to\infty}{\to} \Psi_*(X)e^{i\omega t}$, and $\nu(x, t) \underset{t\to\infty}{\to} \nu_*(x)$, i.e. the flow is stabilized.

5. Let us now find expressions for buoyancy and the moment acting on the wing in the stratified fluid. The pressure gradient on the wing can be calculated from Eq. (2.4), which appears as follows in the coordinate system connected with the wing

$$-\frac{1}{\rho_0}\frac{\partial[p]}{\partial x} = \frac{\partial[u_1]}{\partial t} - V\frac{\partial[u_1]}{\partial x}.$$

Since $[p] = 0$ on the trailing edge, then by using (2.10) we find that

$$[p(x,t)] = \rho_0 V \left([u_1(x,t)] + \frac{1}{V}\frac{\partial}{\partial t}\int_x^c [u_1(\xi,t)]\, d\xi\right). \tag{5.1}$$

Buoyancy and the moment are then found from the formulas

$$Y(t) = -\int_{-c}^{c} [p(x,t)]\, dx; \tag{5.2}$$

$$M(t) = -\int_{-c}^{c} x[p(x,t)]\, dx. \tag{5.3}$$

It follows from (5.1)-(5.2) that for the steady-state motion of a thin wing at a small angle of attack in a weakly stratified fluid, the classical Zhukov formula for buoyancy remains valid:

$$Y = -\rho_0 V\Gamma. \tag{5.4}$$

Inserting Eq. (2.12) for the velocity discontinuity into (5.1)-(5.3), we obtain the following formulas for buoyancy and the moment in the case of harmonic vibrations of the wing:

$$Y = Y_* e^{i\omega t}, \quad M = M_* e^{i\omega t}, \quad æ(\eta, t) = æ_*(\eta)e^{i\omega t};$$

$$Y_* = -2\pi\rho_0 Vc\left[(1 + is)\int_{-1}^{1} æ_*(\eta)\, d\eta + is\int_{-1}^{1} \eta æ_*(\eta)\, d\eta + s_0\int_{-1}^{1} æ_*(\eta)\, d\eta \times\right. \tag{5.5}$$

$$\times \int_0^{\eta+1} e^{-isx} G(s_0 x)\, dx + iss_0 \int_{-1}^{1} æ_*(\eta)\, d\eta \int_0^{\eta+1} (1+\eta-x) e^{-isx} G(s_0 x)\, dx \Big];$$

$$M_* = -2\pi\rho_0 V c^2 \Big[\int_{-1}^{1} \eta æ_*(\eta) d\eta - \frac{is}{2} \int_{-1}^{1} æ_*(\eta)(1-\eta^2)\, d\eta + s_0 \int_{-1}^{1} æ_*(\eta)\, d\eta \times$$
$$\times \int_0^{\eta+1} (\eta - x) e^{-isx} G(s_0 x)\, dx + iss_0 \int_{-1}^{1} \xi\, d\xi \int_{\xi} d\zeta \int_{\zeta} æ_*(\eta) e^{is(\zeta-\eta)} G(s_0(\eta-\zeta))\, d\eta \Big]. \qquad (5.6)$$

Here, s = $\omega c/V$ is the Strouhal number. The first two terms in Eqs. (5.5)-(5.6) are the same as in a uniform fluid, while the last terms determine the effect of stratification. These formulas are also valid for steady-state flow past the wing at $s = 0$.

Let us find the asymptote of buoyancy for the wing during its steady-state motion at a small angle of attack for low values of s_0. Expanding the kernel (3.2) of the integral equation into a series at $z = 0$ and $s_0 << 1$, we have

$$T(\xi, \eta; 0, s_0) = s_0 T_{01} + s_0^2 \ln s_0 T_{02} + \dots,$$

where

$$T_{01} = 1; \qquad T_{02} = -\frac{1}{2}(\eta + 1 - \xi).$$

With motion of the wing at a small angle of attack α, the displacement of the wing is given by the relation $x_2 = \alpha x_1$. Then vertical velocity on the wing

$$f(x_1, t) = \left(\frac{\partial}{\partial t} - V\frac{\partial}{\partial x_1}\right) x_2 = -V\alpha,$$

and this means that

$$f^L(x, p) = -\frac{V\alpha}{p}.$$

Using the method of successive approximation, we find from the Fredholm integral equation that

$$æ = æ_0 + s_0 æ_1 + s_0^2 \ln s_0 æ_2 + \dots,$$

where

$$æ_0(\xi) = -V\alpha\sqrt{\frac{1+\xi}{1-\xi}},$$

while the functions $æ_1$ and $æ_2$ are determined by the formulas

$$æ_1 = -\frac{1}{\pi}\sqrt{\frac{1+\xi}{1-\xi}} \int_{-1}^{1} T_{01}(\xi, \eta) æ_0(\eta)\, d\eta, \quad æ_2 = -\frac{1}{\pi}\sqrt{\frac{1+\xi}{1-\xi}} \int_{-1}^{1} T_{02}(\xi, \eta) æ_0(\eta)\, d\eta.$$

Now inserting these expressions into Eq. (5.5) at $s = 0$, we obtain the asymptote of buoyancy at low s_0:

$$Y = 2\pi\rho_0 c V^2 \alpha (1 - s_0 + s_0^2 \ln s_0). \qquad (5.7)$$

The first term in this formula corresponds to a uniform fluid, while the remaining terms determine the effect of stratification. Since $\ln s_0 < 0$ at $s_0 < 1$, then the following two terms yield a reduction in buoyancy compared to the uniform fluid. This is surprising at first, since denser fluid is located downstream of the wing. However, if stratification is slight, the effect of a

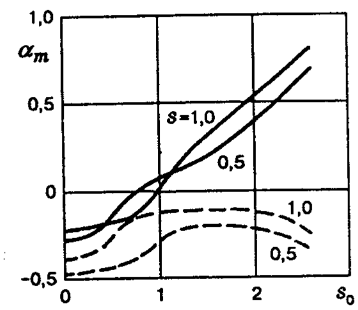

Fig. 8

change in density on the wing turns out to be negligible. As follows from Eq. (5.4), the buoyancy of the wing is determined by the circulation of velocity.

The following physical explanation can be offered for the decrease in circulation around the wing. Figure 2 depicts the flow of a uniform fluid about the wing at an angle of attack. Line l shows the dividing streamline. Not all of the particles can overcome the barrier in a stratified fluid, which causes the dividing streamline to be shifted toward the leading edge. This means that velocity is lower above the wing and greater below the wing than in a uniform fluid. It follows from this that the stratification of density at low s_0 reduces circulation and buoyancy.

6. Let us find a numerical solution to the problem, in addition to the force acting on the wing in the case of steady-state motion at a small angle of attack with harmonic flexural and torsional vibrations of the wing. For the numerical solution, it turns to be more convenient to solve singular equation (2.16) with relations (2.13)-(2.15). In the case of steady-state flow past a wing at an angle of attack, it follows from (2.14)-(2.15) that $\gamma(t) \underset{t\to\infty}{\to}$ Although in this case the velocity discontinuity is equal to zero, i.e., there is no wake, the density of potential does not vanish and is determined by the function

$$h(\xi) = -s_0 \int_{-1}^{1} æ(\eta) e^{is(\xi-\eta)} \left[J_1(s_0(\eta-\xi)) + s_0 \int_{0}^{1+\eta} J_1(s_0(\eta-\xi-x)) G(s_0 x) dx \right] d\eta .$$

For harmonic vibrations of the wing

$$\gamma(t) = \gamma_* e^{i\omega t}, \qquad h(\xi, t) = h_*(\xi) e^{i\omega t},$$

where

$$\gamma_*(t) = -i s_0 \int_{-1}^{1} æ_*(\eta) \left[1 + s_0 \int_{0}^{1+\eta} e^{-isx} G(s_0 x)\, dx \right] d\eta;$$

$$h_*(\xi) = -s_0 \int_{-1}^{1} æ_*(\eta) e^{is(\xi-\eta)} \left[J_1(s_0(\eta-\xi)) + s_0 \int_{0}^{1+\eta} J_1(s_0(\eta-\xi-x)) G(s_0 x)\, dx \right] d\eta .$$

We performed calculations using the discrete vortices method [9] in accordance with the scheme devised by S. M. Belotserovskii: the vortex in the 1/4 segment and the control point in the 3/4 segment from the leading edge. The unknown intensities of the vortices in this case are found from the solution of the corresponding system of linear algebraic equations. We checked the determinant of this system and found it to behave monotonically and smoothly. No discrete eigenvalues were seen at $s_0 < 0$.

Figure 3 shows the results of calculation of the dimensionless force coefficient C_y and moment coefficient C_m in the case of steady-state flow past the wing ($Y = 2\pi\rho_0 c V^2 \alpha C_y$, $M = 2\pi\rho_0 c^2 V^2 \alpha C_m$). As the asymptote, the results of these calculations demonstrate that the buoyancy of the wing decreases at low values of s_0. The reduction is quite large and can reach roughly 0.6 of the corresponding value in a uniform fluid. Buoyancy then increases at $s_0 > 1$, and at $s_0 > 2$ buoyancy in the stratified fluid becomes greater than in the uniform fluid. This nonmonotonic behavior is evident from Eq. (5.7) for buoyancy.

At $s_0 > 1$, the last terms of this equation become dominant and produce an increase in buoyancy. This phenomenon is probably connected physically with effects downstream at $s_0 > 1$ [10]. The fluid ahead of the wing stagnates, the velocity discontinuity on the wing begins to increase, and there is a corresponding increase in circulation and buoyancy. The fact that the force coefficient C_y increases without bound at $s_0 \to \infty$ can be attributed to the chosen normalization, which is traditional in airfoil theory. If we change over to dimensional variables, then buoyancy Y approaches zero as $V \to 0$.

It is evident from Fig. 3 that the moment of the force also approaches zero, i.e. the center of pressure is shifted toward the middle of the wing. This is explained by the the redistribution of fluid velocity around the wing.

Figure 4 shows graphs of the velocity gradient on the wing for different values of s_0. It is apparent that monotonicity is disturbed with an increase in s_0 and a "hump" appears at the middle of the wing. It is known that singular integral equations on the segment have three types of solutions: two solutions not bounded on either end; two solutions bounded on one end; a third solution bounded on both ends if an additional condition is satisfied on the right side. At $s_0 \to \infty$, there will likely be a transition from a solution not bounded on the leading edge to a solution bounded on both ends with a critical point on the leading edge.

We also performed calculations of the transient forces and moments in the case of flexural and torsional vibrations of the wing. Figures 5-8 show the dependence of the modulus and phase of the dimensionless force and moment coefficients on the parameter s_0 for Strouhal numbers of 0.5 and 1 (the solid and dashed lines correspond to flexural and torsional vibrations, respectively). Here, we also note that the force modulus is nonmonotonic and that the moment decreases with an increase in the parameter s_0, i.e., with a decrease in the Froude number. Thus, the effect of stratification on the hydrodynamic characteristics of a wing is substantial.

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